International Journal on Emerging Technologies **11**(3): 755-758(2020)

ISSN No. (Print): 0975-8364
ISSN No. (Online): 2249-3255

New Optimal Fourth Order Iterative Method for Solving Nonlinear Equations

Ekta Sharma[1], Sunil Panday[2] and Mona Dwivedi[3]
[1]Research Scholar, Department of Mathematics, National Institute of Technology (Manipur), India.
[2]Assistant Professor, Department of Mathematics, National Institute of Technology (Manipur), India.
[3]Researcher, Department of Mathematics, MANIT Bhopal (Madhya Pradesh), India.

(Corresponding author: Ekta Sharma)
(Received 10 March 2020, Revised 06 May 2020, Accepted 08 May 2020)
(Published by Research Trend, Website: www.researchtrend.net)

ABSTRACT: We have presented a new optimal fourth order iterative method in this paper. Every iteration desires one function evaluation and two first derivative evaluations and therefore the efficiency of this method is 1.5874. Many researchers have generated several order techniques, whenever the second and higher order derivatives of the function exist in a neighbourhood of the root. But the cost of evaluating the second derivative of the function is itself a cumbersome problem. In this paper, we derive a high-order iteration technique to discover the root of nonlinear equations at the low computational cost of the first derivative of the function. In fact, we have obtained the optimal order of convergence which satisfies the Kung and Traub optimality conjecture. Kung and Traub conjectured that the multipoint iteration method without memory based on n-evaluations could achieve optimal convergence order 2^{n-1}. The analysis of convergence shows that the new technique has fourth order convergence. In addition, the theoretical convergence property of our technique is fully explored with the help of main theorem that declares the convergence order. Numerical comparisons with some well-known schemes having fourth order of convergence are presented with several examples to verify the efficiency of present technique. These results declare the performance of our technique. Finally, the proposed method is found to be more efficient as compare to some standard iterative methods of same order.

Keywords: Efficiency index, Kung-Traub conjecture, Iterative method, Non-linear equations, Newton's method, Order of convergence.

Abbreviations: FSM, fazlollah soleymani method; RSM, Rajni Sharma method; SKKM, Sanjay Kumar Khattri method; NPM, new proposed method.

I. INTRODUCTION

The recent advancements in numerical analysis have introduced to the scientific community numerous methods that have made iterative calculations simple and highly accurate. Most of these methods are computational and thus have reduced manual effort to a great deal. We are much empowered to create advanced methods in the current era with the help of digital computer, technological interventions and advanced arithmetical computation in the spacious literature [1].

In this paper, we present a new iterative method for deducing a simple root x of a non-linear equation

$$g(x)=0 \tag{1}$$

Where $g : X \subset R \to R$ is a scalar function on an open interval x.

The classical Newton method is one of the well-recognized methods for finding roots of nonlinear equations [2] written by

$$x_{n+1}=x_n-\frac{g(x_n)}{g'(x_n)},\quad n=0,1,2 \tag{2}$$

The Newton method converges quadratically [2] in some neighborhoods of a simple root $x=r$ of $g(x) = 0$. In contemporary times, the efficiency of an iterative method is calculated through efficiency index by $p^{1/m}$, where p is the order of convergence and m is the total number of function evaluations per iteration [3, 4]. For systematizing classifying iterative methods, the first efforts were made by Traub [2]. The Kung and Traub conjecture [5] that an iterative method without memory with m-function evaluation per iteration function for finding roots of a nonlinear equation, could achieve optimal order $p = 2^{m-1}$. In recent years however, many other authors have frequently brought up higher order iterative methods and investigated the convergence analysis of them for finding non-linear equations. Behl *et al.*, (2019) derived highly efficient family of iterative methods with sixth order convergence [6], Chun (2008) proposed some fourth-order iterative methods for solving non-linear equations [7]. Kung and Traub derived optimal order of one point and multipoint iteration [5], Nouri *et al.*, (2019) suggested two high order iterative methods including five and ten orders convergence [8]. Kou *et al.,* (2010) proposed some eighth order root finding three-step methods [9], Sen *et al.,* (2012) derived computational pitfalls of high-order methods for non-linear equations [10]. Khattri *et al.*, (2011) proposed unifying fourth-order family of iterative methods [11]. Wang and Tao derived a new newton method with memory for solving nonlinear equations [12]. Chauhan and Vikanshi, (2019) suggested Convergence of CUIA Iteration in Real Hilbert Space [13]. Villafuerte *et al.,* (2019) suggested an iterative method to solve non-linear equations [14] and the

references therein. General books on this subject are Iterative Methods for the Solution of Equation [2] and Numerical Analysis: An Introduction [15]. In the spacious literature, we can lightly present a development of many iterative methods to increasing order of convergence for finding ideas for non-linear equation [16-22]. In this work we developed iterative method of optimal fourth order without memory, which clearly indicates Kung-Traub conjecture with three function evaluation.
The rest of the paper is organized in below order: In the next section, we will develop the new iterative method of optimal order four which requires three function evaluations per iteration. We also provide the proof of the theorem for convergence order of the proposed scheme in this section. Section III presents a numerical comparison between the proposed method and other existing fourth order methods. Finally, some concluding remarks of the paper are in section IV.

II. DEVELOPMENT OF SCHEME

In this section, we present a new optimal fourth order iterative method as follows:

$$x_{n+1} = x_n - \frac{4g(x_n)}{g'(x_n)+3g'(y_n)}(1+u_n{}^3) - \frac{9}{16}\left(\frac{\phi}{g'(x_n)}\right)^2 u_n{}^3 \quad (3)$$

where,

$$u_n = \frac{g(x_n)}{g'(x_n)} \quad (4)$$

$$y_n = x_n - \frac{2}{3}u_n \quad (5)$$

$$\phi = \frac{g'(x_n) - g'(y_n)}{u_n} \quad (6)$$

If x_0 is some initial approximation and it is assumed that the denominator of Eqn. (3) is not equal to zero, then the above method has convergence order 4 in some neighborhood of simple root *r*.

III. THEOREM

Let $g: X \subset R \to R$ be a sufficiently differentiable function in an open interval X and $r \in X$ be a simple zero of $g(x)=0$. If x_0 is in the neighbourhood of r, then the method given by Eqn. (3) has fourth order convergence with the following error equation

$$e_{n+1} = (-1+4t_2^3 - t_2t_3 + \frac{t_4}{9})e_n^4 + (3t_2 - 27t_2^4 + 26t_2^2t_3 - 2t_3^2 - \frac{20t_2t_4}{9} + \frac{8t_5}{27})e_n^5 + O(e_n^6) \quad (7)$$

Where,

$$e_n = x_n - r \text{ and } t_i = \frac{g^n(r)}{g'(r)},\ j \geq 2$$

Proof: If x_n be the n^{th} approximation to the root `r' of equation (1), then the Taylor's expansion of $g(x_n)$ about r can be written as

$$g(x_n) = g(r) + e_n g'(r) + \frac{(e_n)^2}{2!}g''(r) + \frac{(e_n)^3}{3!}g'''(r) + \ldots \quad (8)$$

Substituting *g*(*r*) = 0 and simplifying, we have

$$g(x_n) = g'(r)[e_n + t_2e_n^2 + t_3e_n^3 + t_4e_n^4 + t_5e_n^5 + O(e_n^6)] \quad (9)$$

Where,

$$t_i = \frac{g^j(r)}{j!\,g'(r)} \text{ for } j = 2,3,4\ldots.$$

Furthermore, we get

$$g'(x_n) = g'(r)[1 + 2t_2e_n + 3t_3e_n^2 + 4t_4e_n^3 + 5t_5e_n^4 + 6t_6e_n^5 + O(e_n^6)] \quad (10)$$

Substituting these values in Eqn. (4), then we will get

$$u_n = \frac{g(x_n)}{g'(x_n)} = e_n - t_2e_n^2 + (2t_2^2 - 2t_3)e_n^3 + (-4t_2^3 + 7t_2t_3 - 34t)e_n^4 + (8t_2^4 - 20t_2^2t_3 + 6t_3^2 + 10t_2t_4 - 4t_5)e_n^5 + O(e_n^6) \quad (11)$$

and hence, using Eqn. (5) and Eqn. (11), we obtain

$$y_n = x - \frac{2}{3}u_n = \frac{e_n}{3} + \frac{2t_2e_n^2}{3} - \frac{4}{3}(t_2^2 - t_3)e_n^3 + \frac{2}{3}(4t_2^3 - 7t_2t_3 + 3t_4)e_n^4 - \frac{4}{3}(4t_2^4 - 10t_2^2t_3 + 3t_3^2 + 5t_2t^4 - 2t_5)e_n^5 + O(e_n^6) \quad (12)$$

The expansion of $g'(x_n)$ about r is given as

$$g'(y_n) = g'(r)[1 + \frac{2t_2}{3} + \frac{1}{3}(4t_2^2 + t_3)e_n^2 + \frac{4}{27}(-18t_2^3 + 27t_2t_3 + t_4)e_n^3 + \frac{1}{81}(216(2t_2^4 - 4t_2^2t_3 + t_3^2) + 396t_2t_4 + 5t_5)e_n^4 + \frac{4}{81}(-9(24t_2^5 - 60t_2^3t_3 + 27t_2t_3^2 + 30t_2^2t_4 - 13t_3t_4) + 118t_2t_5)e_n^5 + O(e_n^6)] \quad (13)$$

The denominator of Eqn. (3) is given as

$$g'(x) + 3g'(y) = g'(r)[4 + 4t_2e_n + 4(t_2^2 + t_3)e_n^2 + (-8t_2^3 + 12t_2t_3 + \frac{40t_4}{9})e_n^3 + \frac{4}{27}(54(2t_2^4 - 4t_2^2t_3 + t_3^2) + 99t_2t_4 + 35t_5)e_n^4 + (-32t_2^5 + 80t_2^3t_3 - 36t_2t_3^2 - 40t_2^2t_4 + \frac{52t_3t_4}{3} + \frac{472t_2t_5}{27})e_n^5 + O(e_n^6)] \quad (14)$$

Then, using Eqns. (9) , (10) and (13), we will obtain

$$\frac{4g(x)}{g'(x) + 3g'(y)} = e_n - t_2^2e_n^3 + (3t_2^3 - 3t_2t_3 - \frac{t_4}{9})e_n^4 + (-6t_2^4 + 12t_2^2t_3 - 2t_3^2 - \frac{32t_2t_4}{9} - \frac{8t_5}{27})e_n^5 + O(e_n^6) \quad (15)$$

and we will get the equation

$$\frac{4g(x)}{g'(x)+3g'(y)}(1+u_n^3) = e_n - t_2^2e_n^3 + (1 + 3t_2^3 - 3t_2t_3 - \frac{t_4}{9})e_n^4 + (-6t_2^4 + 12t_2^2t_3 - 2t_3^2 + t_2(-3 - \frac{32t_4}{9}) - \frac{8t_5}{27})e_n^5 + O(e_n^6) \quad (16)$$

Furthermore, substituting values in Eqn. (6) from Eqns. (10) (11) and (13), we will have

$$\phi = \frac{g'(x)-g'(y)}{u_n} = g'(r)\Big[\frac{4t_2}{3}+\frac{8t_3e_n}{3}+\frac{4}{27}(9t_2t_3+26t_4)e_n^2+$$
$$\frac{8}{81}(27t_3(-t_2^2+t3)+30t_2t_4+50t_5)e_n^3+$$
$$\frac{4}{27}(36t_2^3t_3-63t_2t_3^2-32t_2^2t_4+67t_3t_4+30t_2t_5)e_n^4+O(e_n^5)\Big] \quad (17)$$

Substituting all above values in equation (3) and we get the error equation

$$e_{n+1} = (-1+4t_2^3-t_2t_3+\frac{t_4}{9})e_n^4+(3t_2-27t_2^4+26t_2^2t_3-$$
$$2t_3^2-\frac{20t_2t_4}{9}+\frac{8t_5}{27})e^5+O(e_n^6) \quad (18)$$

IV. RESULTS AND DISCUSSION

We present some test functions with roots which are available with initial guesses in Table 1. We compare some numerical test results to illustrate the efficiency of the new iterative method in Table 2. We compare new proposed method (NPM) given by equation (3) with the three existing iterative methods of order four. Before doing the comparisons, we need to review some of the most important fourth order methods, which are available in the literature.

Khattri *et al.,* (2011) presented a fourth order iterative method [SKKM] [11]. The method of the family are fourth order convergent and require only three evaluations during each iteration:

$$y_n = x_n - \frac{g(x_n)}{g'(x_n)} \quad (19)$$

$$x_{n+1} = x_n - \frac{1}{g'(x_n)}\Big[\frac{2g(x_n)^2-3g(x_n)g(y_n)-g(y_n)^2}{2g(x_n)-5g(y_n)}\Big] \quad (20)$$

Soleymani (2011) [23] suggested optimal fourth order iterative method [FSM] free from derivative, which includes three evaluations of the function per iteration and free from any derivative calculation:

$$y_n = x_n - \frac{g(x_n)}{g[x_n, A_n]} \quad (21)$$

$$x_{n+1} = y_n - \frac{(A_n-y_n)g(y_n)\Big[1+2\frac{g(y_n)}{g(A_n)}\Big]}{(x_n-y_n)g[x_n,A_n]+(A_n-x_n)g[x_n,y_n]} \quad (22)$$

Where, $A_n = x_n + g(x_n)$ and $n = 0,1,2,...$

Sharma and Bahl (2015) [24] proposed the two step weighted Newton iteration method [RSM] which is optimal fourth order of the type

$$y_k = x_k - \frac{2}{3}\frac{g(x_k)}{g'(x_k)}, \quad (23)$$

$$x_{k+1} = x_k - \Big(-\frac{1}{2}+\frac{9}{8}\frac{g'(x_k)}{g'(y_k)}+\frac{3}{8}\frac{g'(y_k)}{g'(x_k)}\Big)\frac{g(x_k)}{g'(x_k)} \quad (24)$$

All computations are performed by using MATHEMATICA 12.0 for the several test functions:

Table 1: Test functions and their roots.

Non-linear function	Roots	Initial guesses
$g_1(x) = (x-1)^3 - 1$	0.00003636485591	2.7
$g_2(x) = x^2 - e^x - 3x + 2$	47	3.5
$g_3(x) = x^3 + x^2 - 2$	35	2.0
$g_4(x) = x^3 - e^{-x}$	45	3.0
$g_5(x) = e^x - 3x^2$	3.733079	5.0

Table 2: Comparison of various iterative methods.

g(x)	FSM	RSM	SKKM	NPM
$g_1(x)$	5.4275×10^{-17}	4.3356×10^{-73}	8.4338×10^{-66}	1.3839×10^{-96}
$g_2(x)$	2.1820×10^{-24}	2.1056×10^{-39}	1.0817×10^{-39}	3.6335×10^{-43}
$g_3(x)$	1.2094×10^{-10}	5.1446×10^{-65}	2.5050×10^{-58}	3.1481×10^{-70}
$g_4(x)$	9.8339×10^{-81}	5.8149×10^{-168}	1.2338×10^{-154}	6.4826×10^{-169}
$g_5(x)$	1.4226×10^{-92}	1.4786×10^{-61}	1.2232×10^{-32}	1.0853×10^{-319}

V. CONCLUSION

The new optimal fourth order iterative method is a novel procedure that we have devised and introduced in this paper. The primary idea of this method is to achieve successfully higher order convergence.

Each iteration requires one function evaluation and two first derivative evaluations. Comparisons of the proposed method have been made with other existing method and it was seen that our method provides equal and better performance of the others.

VI. FUTURE SCOPE

The proposed work can be used for develop iterative methods for multiple roots and construct some new methods for solving system of non-linear equations. In future, we intend to develop the higher order convergence method in with memory version using self-accelerating parameters.

Conflict of Interest. The authors, Ekta Sharma, Sunil Panday and Mona Dwivedi, declare no conflict of interest associated with this work.

ACKNOWLEDGEMENTS

The authors are grateful to the reviewers and editor for their constructive comments and insightful suggestions, which led to the significant improvement of the paper.

REFERENCES

[1]. Sharma, J. R., & Arora, H. (2016). A new family of optimal eighth order methods with dynamics for nonlinear equations. *Applied Mathematics and Computation*, *273*(C), 924-933.

[2]. Traub, J. F. (1964). Iterative Methods for the Solution of Equation. *Prentice-Hall. Engle-wood Cliffs, NJ. USA.*

[3]. Ostroswski, A. M. (1960). Solution of Equations and Systems of Equations. *Academic Press, New York, NY, USA.*

[4]. Soleymani, F., & Vanani, S. K. (2011). Optimal Steffensen-type methods with eight order of convergence. *Computers and Mathematics with Applications*, *62*, 4619-4626.

[5]. Kung, H. T., & Traub, J. F. (1974). Optimal Order of One point and Multipoint Iteration. *Applied Mathematics and Computation, 21*(4), 643-651.

[6]. Behl, I. R., Sarria, I., Gonzalez, R., & Magrenan, A. A. (2019). Highly Efficient Family of Iterative Methods for Solving Non-linear Models. *Journal of Computational and Applied Mathematics*, *346*, 110-132.

[7]. Chun, C. (2008). Some Fourth-Order Iterative Methods for Solving Non-linear Equations. *Appl. Math. Comput., 195*(2), 454-459.

[8]. Nouri, K., Ranjbar, H., & Torkzadeh, L. (2019). Two High Order Iterative Methods for Roots of Non-linear Equations. *Journal of Mathematics*, *51*(3), 47-59.

[9]. Kou, J., Wang, X., & Li, Y. (2010). Some Eighth Order Root Finding Three-step Methods. *Commun Non-linear Sci Numer Simulat*, *15*(3), 536-544.

[10]. Sen, S. K., Agarwal, R. P., & Khattri, S. K. (2012). Computational Pitfalls of High-Order Methods for Non-linear Equations. *J. Appl. Math. Inform.*, *30*, 395-411.

[11]. Khattri, S. K., Noor, M. A., & Said E. Al. (2011). Unifying Fourth-Order Family of Iterative Methods. *Appl. Math. Lett.*, *24*(8), 1295-1300.

[12]. Wang, X., & Tao, Y. (2020). A New Newton Method with Memory for Solving Nonlinear Equations. *Mathematics*.

[13]. Chauhan, S. S., & Vikanshi. (2019). Convergence of Iteration in Real Hilbert Space. *International Journal on Emerging Technologies, 10*(2b), 188-192.

[14]. Villafuerte, R., Medina, J., Ruben, A., Villafuerte, S., & Juarez, V. (2019). An Iterative Method to Solve Non-linear Equations. *Univarsal Journal of Electrical and Electronic Engineering, 6*(1), 14-22.

[15]. Gautschi, W. (1997). Numerical Analysis: An Introduction, Birkhauser.

[16]. Chun, C. (2006). A New Iterative Method for Solving Non-linear Equations. *Appl. Math. Comput.*, *178*, 415-422.

[17]. Thukral, R. (2008). Introduction to a Newton-Type Method for Solving Non-linear Equations. *Appl. Math. Comput.*, *195*(2), 663-0668.

[18]. Liu, L., & Wang, X. (2010). Eighth-Order Methods with High Efficiency Index for Solving Non-linear Equations. *Applied Mathematics and Computation*, *215,* 3449-3454.

[19]. Soleymani, F., Khattri, S. K., & Vanani, S. K. (2012). Two New Classes of Optimal Jarratt-Type Forth-Order Methods. *Appl. Math. Lett., 25*(5), 847-853.

[20]. Mirzaee, F., & Hamzeh, A. (2014). A Sixth Orther Method for Solving Non-linear Equations. *International Journal of Mathematical Modelling & Computations, 4*(1), 53-58.

[21]. Al-Oufi, A. M. & Al-Subaihi, I. A. (2018). An Optimal Eifht-Order of King-Type Family Iterative Method For Solving Non-linear Equations. *Journal of Research in Applied Mathematics, 5*(1), 16-21.

[22]. Thukral, R. (2018). New Fifth-Order Simpson-Type Method for Solving Nonlinear Equations. *American Journal of Computational and Applied Mathematics, 8*(4), 80-84.

[23]. Soleymani, F. (2011). Optimal Fourth Order iterative method free from derivative. *Miskole Mathematical Notes*, *12*(2), 255-264.

[24]. Sharma, R., & Bahl, A. (2015). An Optimal Fourth Order Iterative method for solving Nonlinear Equations and Dynamics. *Hindawi Pubishing Corporation Journal of Complex Analysis*.

How to cite this article: Sharma, E., Panday, S. and Dwivedi, M. (2020). New Optimal Fourth Order Iterative Method for Solving Nonlinear Equations. *International Journal on Emerging Technologies*, *11*(3): 755–758.